

A Weak Convergence Criterion Constructing Changes of Measure*

Jose Blanchet[†]

Department of Operations Research and Industrial Engineering
Columbia University

Johannes Ruf[‡]

Oxford-Man Institute of Quantitative Finance and Department of Mathematics
University of Oxford

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Abstract

We provide a necessary and sufficient condition, based on weak convergence considerations, in order to guarantee that a nonnegative local martingale is indeed a martingale. Typically, conditions of this sort are expressed in terms of integrability conditions (such as the well-known Novikov condition). The weak convergence approach that we propose allows to replace integrability conditions by a suitable tightness condition. We then provide several applications of this approach; in particular, we show that the square-integrability of a certain functional applied to the solution of a stochastic differential equation is sufficient for the martingality of a certain local martingale.

1 Introduction

Changing the probability measure is a powerful tool in modern probability. Changes of measure arise in areas of wide applicability such as in mathematical finance, in the setting of so-called equivalent pricing measures. A change of probability measure often relies on the specification of a nonnegative martingale process which in turn yields the underlying Radon-Nikodym derivative behind the change of measure.

The key step in the typical construction of changes of measure involves showing the martingale property of a process of putative Radon-Nikodym derivatives. In order to verify this martingale property one often starts by defining a process that easily can be seen to be a local martingale. This is the standard situation, for example, in changes of measure for diffusion processes; in this framework, a standard application of Itô's formula guarantees that a candidate exponential process is a local martingale. The difficult part then involves ensuring that the local martingale is actually a martingale.

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[†]E-Mail: jose.blanchet@columbia.edu

[‡]E-Mail: johannes.ruf@oxford-man.ox.ac.uk

Since the distinction between local martingales and martingales involves verification of integrability properties (the ones behind the strict definition of a martingale), it is most natural to search for a criterion based on integrability of the underlying local martingale. This is the basis, for instance, of the so-called Novikov's condition, which is a well-known criterion used to verify the martingale property of an exponential local martingale in the diffusion setting. Nevertheless, if ultimately one has the existence of a new probability measure, then one has a martingale defined by the corresponding change of measure. Thus, it appears that lifting the local martingale property for a nonnegative stochastic process to a bona-fide martingale property has more to do with the fact that the induced probability measure is indeed well-defined.

Our contribution in this note consists in connecting tightness with the verification of the martingale property of a positive local martingale. We discuss the construction of approximating martingale processes which allow us to define a sequence of probability measures. We show that if the approximating martingale processes (which are no longer martingales under the induced changes of measure) are tight under the induced changes of measure and under the topology generated by convergence of the finite dimensional coordinates, then the original local martingale process is indeed a martingale.

It then is important to note that showing the martingale property of the underlying positive local martingale becomes an exercise in tightness in a very weak topology. Given the enormous literature in weak convergence analysis of stochastic processes, we feel that our test of martingality would be a useful one. For example, in order to show tightness of a sequence of random variables $\{A_n\}_{n \in \mathbb{N}}$ of the form $A_n = \exp(B_n + C_n)$ it is sufficient to show tightness for the sequences of random variables $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$ separately; a task that is often easy, as we shall illustrate. In addition, the martingale property of a natural approximation to the local martingale process of interest is usually immediately seen to be a martingale. In order to illustrate these observations we study a few general applications in Section 3.

Let us provide a precise mathematical statement of our contribution. For the sake of clear notation, for a sequence of random variables $\{Y_n\}_{n \in \mathbb{N}}$, each defined on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$, and a random variable Y , defined on a probability space (Ω, \mathcal{F}, P) , we write $(P_n, Y_n) \xrightarrow{w} (P, Y)$ if $\lim_{n \uparrow \infty} P_n(Y_n \leq x) = P(Y \leq x)$ for each continuity point x of $P(Y \leq \cdot)$. Our main basic result is given next (a slight generalization is provided in Section 2):

Theorem 1. *Let $M = \{M(t)\}_{t \geq 0}$ denote a nonnegative sub- or supermartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, P)$ with corresponding expectation operator E , and let $\{M_n\}_{n \in \mathbb{N}}$ with $M_n = \{M_n(t)\}_{t \geq 0}$ denote a sequence of nonnegative martingales, each defined on a filtered probability space $(\Omega_n, \mathcal{F}_n, \{\mathcal{F}_n(t)\}_{t \geq 0}, P_n)$ with corresponding expectation operators E_n such that $M_n(0) = 1$.*

1. *Fix a sequence of (deterministic) times $\{t_m\}_{m \in \mathbb{N}}$ with $t_1 = 0$ and $\lim_{m \uparrow \infty} t_m = \infty$ and assume that $(P_n, M_n(t_m)) \xrightarrow{w} (P, M(t_m))$ for each $m \in \mathbb{N}$. Then M is a true martingale with $M(0) = 1$ if and only if*

$$\sup_{n \in \mathbb{N}} Q_n^m(M_n(t_m) \geq \kappa) \rightarrow 0 \quad (1)$$

as $\kappa \uparrow \infty$ for each $m \in \mathbb{N}$, where the probability measures Q_n^m are defined via $dQ_n^m = M_n(t_m)dP_n$. That is, M is a true martingale if and only if $\{M_n(t_m)\}_{n \in \mathbb{N}}$ is tight under the sequence of measures $\{Q_n^m\}_{n \in \mathbb{N}}$ for each $m \in \mathbb{N}$.

2. Assume that $M(\infty) = \lim_{t \uparrow \infty} M(t)$ exists, $\lim_{n \uparrow \infty} E[M_n(\infty)] = 1 = M(0)$, and $(P_n, M_n(\infty)) \xrightarrow{w} (P, M(\infty))$. Then M is a uniformly integrable martingale if and only if

$$\sup_{n \in \mathbb{N}} E_n [M_n(\infty) \mathbf{1}_{\{M_n(\infty) \geq \kappa\}}] \rightarrow 0$$

as $\kappa \uparrow \infty$. More specifically, if the M_n 's are uniformly integrable martingales and define probability measures Q_n^∞ via $dQ_n^\infty = M_n(\infty)dP_n$, then M is a uniformly integrable martingale if and only if

$$\sup_{n \in \mathbb{N}} Q_n^\infty(M_n(\infty) \geq \kappa) \rightarrow 0$$

as $\kappa \uparrow \infty$.

The proof of Theorem 1 is extremely simple and only relies on the definition of tightness. Nevertheless, as we shall illustrate in our applications we are convinced that the point of view taken in Theorem 1 is useful as it provides a simple and standard approach to prove the martingality of a nonnegative local martingale.

Relevant literature

The standard way to show martingality of a nonnegative local martingale is to check some standard integrability condition; see for example Novikov (1972), Kazamaki and Sekiguchi (1983), or Ruf (2012). If the local martingale dynamics include jumps, a case that we explicitly allow here, then integrability conditions exist but they might not be trivial to check; see Protter and Shimbo (2008) for such conditions and related literature.

Under additional assumptions on the local martingale, such as the assumption that it is constructed via an underlying Markovian process, further sufficient (and sometimes also necessary) criteria can be derived. Here we only provide the reader with some pointers to this vast literature. The following papers develop conditions different from Novikov-type conditions by utilizing the (assumed) Markovian structure of some underlying stochastic process, and contain a far more complete list of references: Cheridito et al. (2005), Blei and Engelbert (2009), and Mijatović and Urusov (2012). Kallsen and Muhle-Karbe (2010) study the martingality of stochastic exponentials of affine processes; their approach via the explicit construction of a candidate measure and the use of a simple lemma in Jacod and Shiryaev (2003) is close in spirit to our approach.

The weak existence of solutions to stochastic differential equations is often proven by means of changing the probability measure, see for example Portenko (1975), Engelbert and Schmidt (1984), Yan (1988), or Stummer (1993). This strategy for proving the weak existence of solutions requires the true martingality of the putative Radon-Nikodym density. Our approach to prove martingality of such a density is in the spirit of the reverse direction: The tightness condition that implies the martingality of a putative Radon-Nikodym density by Theorem 1 corresponds basically to the asserted existence of a certain probability measure – often corresponding to the existence of a solution to a stochastic differential equation.

2 Martingality and tightness

In this section, we prove Theorem 1 and make some related observations. The proof of Theorem 1 relies on the following simple but powerful result:

Proposition 1. Let Y denote a nonnegative random variable defined on (Ω, \mathcal{F}, P) with corresponding expectation operator E , and let $\{Y_n\}_{n \in \mathbb{N}}$ denote a sequence of integrable, nonnegative random variables defined on $(\Omega_n, \mathcal{F}_n, P_n)$ with corresponding expectation operators E_n such that $(P_n, Y_n) \xrightarrow{w} (P, Y)$ and $\lim_{n \uparrow \infty} E_n[Y_n] = 1$. Then, $E[Y] = 1$ if and only if

$$\sup_{n \in \mathbb{N}} E_n[Y_n \mathbf{1}_{\{Y_n \geq \kappa\}}] \rightarrow 0$$

as $\kappa \uparrow \infty$.

Proof. Assume that $E[Y] = 1$. Then, for fixed $\kappa > 1$ and for a continuous function $f : [0, \infty) \rightarrow [0, \kappa]$ with $f(x) \leq x$ for all $x \geq 0$, $f(x) = x$ for all $x \in [0, \kappa - 1]$ and $f(x) = 0$ for all $x \in [\kappa, \infty)$, we compute that

$$\begin{aligned} E_n[Y_n \mathbf{1}_{\{Y_n \geq \kappa\}}] &= E_n[Y_n] - E_n[Y_n \mathbf{1}_{\{Y_n < \kappa\}}] \leq E_n[Y_n] - E_n[f(Y_n)] \\ &\rightarrow 1 - E[f(Y)] \leq 1 - E[Y \mathbf{1}_{\{Y \leq \kappa - 1\}}] = E[Y \mathbf{1}_{\{Y > \kappa - 1\}}] \end{aligned}$$

as $n \uparrow \infty$. As $E[Y \mathbf{1}_{\{Y > \kappa - 1\}}]$ can be made arbitrarily small by increasing κ (because Y is integrable by assumption), we obtain one direction of the statement. For the other direction, fix $\epsilon > 0$ and the continuous, bounded function $f : [0, \infty) \rightarrow \mathbb{R}$ with $f(x) = x \wedge \kappa$ for all $x \geq 0$. Then

$$E[Y] \geq E[f(Y)] = \lim_{n \uparrow \infty} E_n[f(Y_n)] \geq \liminf_{n \uparrow \infty} E_n[Y_n \mathbf{1}_{\{Y_n < \kappa\}}] = \liminf_{n \uparrow \infty} (E_n[Y_n] - E_n[Y_n \mathbf{1}_{\{Y_n \geq \kappa\}}]) \geq 1 - \epsilon$$

for κ large enough. This yields $E[Y] = 1$. \square

We are now ready to prove Theorem 1:

Proof of Theorem 1. For the first statement, observe that (1) and the martingality of all processes M_n imply that $E[M(t_m)] = 1$ for all $m \in \mathbb{N}$ by Proposition 1. Since $E[M(t)]$ is assumed to be monotone in t , this yields the martingality of M . The reverse direction is a direct application of the same proposition. For the second statement, observe that the uniform integrability of M is equivalent to $E[M(\infty)] = 1$. Proposition 1 then yields the result. \square

A look at its proof yields that the statement of Theorem 1 can be further generalized since for each t_k a different approximating sequence of true martingales might be used.

The following corollary can be interpreted as a generalization of Theorem 1.3.5 in Stroock and Varadhan (2006) to processes with jumps. See also Lemma III.3.3 in Jacod and Shiryaev (2003) for a similar statement where a certain candidate measure Q is assumed to exist. We remark that the sequence of stopping times in the statement could, but need not, be a localization sequence of a local martingale; for example, it is sufficient that the stopping times converge to the first hitting time that the local martingale hits zero.

Corollary 1. Let $\{\tau_n\}_{n \in \mathbb{N}}$ be a sequence of stopping times and $M_n \equiv M^{\tau_n}$ the stopped versions of a nonnegative local martingale M with $M(0) = 1$. Assume that $M_n(t) \rightarrow M(t)$ P -a.s. as $n \uparrow \infty$ for all $t > 0$ and that M_n is a true martingale. Further, define $dQ_n = M_n dP$. Under these assumptions the following statements hold: If $Q_n(\tau_n \leq t) \rightarrow 0$ as $n \uparrow \infty$ for all $t > 0$, then M is a martingale. Further, under the additional assumption that $\tau_n \rightarrow \infty$ P -a.s. as $n \uparrow \infty$, the converse also holds; that is, if M is a martingale then $Q_n(\tau_n \leq t) \rightarrow 0$ as $n \uparrow \infty$ for all $t > 0$.

Proof. Fix $t, \kappa > 0$ and $n \in \mathbb{N}$ and observe that $(P, M_n(t)) \xrightarrow{w} (P, M(t))$ as well as

$$Q_n(M_n(t) > \kappa) \leq Q_n(\tau_n \leq t) + E[M_n(t)\mathbf{1}_{\{\tau_n > t\}} \cap \{M_n(t) > \kappa\}] \leq Q_n(\tau_n \leq t) + E[M(t)\mathbf{1}_{\{M(t) > \kappa\}}]$$

because $M_n(t)\mathbf{1}_{\{\tau_n > t\}} = M(t)\mathbf{1}_{\{\tau_n > t\}}$. Since by assumption we can make the first term on the right-hand side arbitrarily small by increasing n , the martingality of M follows directly from dominated convergence and Theorem 1. For the reverse direction, assume that M is a martingale and that $\tau_n \rightarrow \infty$ P -a.s. as $n \uparrow \infty$. Then, $Q_n(\tau_n \leq t) = E[M_n(t)\mathbf{1}_{\{\tau_n \leq t\}}] = E[M(t)\mathbf{1}_{\{\tau_n \leq t\}}] \rightarrow 0$ as $n \uparrow \infty$ by dominated convergence. \square

The next result is of course well-known and only a very special case of, for instance, the theory of BMO martingales; see for example Kazamaki (1994). However, as we shall use the result below and as we would like to make this note self-contained, we provide a proof based on the observations we have made here before:

Corollary 2. *Let $L = \{L(t)\}_{t \geq 0}$ denote a continuous local martingale on some probability space. Assume there exists some increasing function $c : [0, \infty) \rightarrow \mathbb{R}$ such that $P(A_t \cup B_t) = 1$ for all $t \geq 0$, where $A_t := \{L(t) \leq c(t)\}$ and $B_t := \{\langle L \rangle(t) \leq c(t)\}$. Then $M = \mathcal{E}(L)$ is a martingale.*

Proof. Let $\{\tau_n\}_{n \in \mathbb{N}}$ denote the first hitting times of integers by M and fix $t > 0$. Obviously, $M_n \equiv M^{\tau_n}$ satisfies $M_n(t) \rightarrow M(t)$ P -a.s. as $n \uparrow \infty$. With the probability measures Q_n as defined in Corollary 1, we observe that $\{Q_n(\tau_n \leq t)\}_{n \in \mathbb{N}}$ is a decreasing sequence. Now, fix $\epsilon \in (0, 1)$ and some $n_0 \in \mathbb{N}$ with $n_0 > \exp(c(t))/\epsilon$ and observe that

$$\{\tau_{n_0} \leq t\} = \{M^{\tau_{n_0}}(t) \geq n_0\} = \{M^{\tau_{n_0}}(t) \geq n_0\} \cap B_t \subset \left\{ \widetilde{M}^{\tau_{n_0}}(t) > \frac{1}{\epsilon} \right\}$$

P -a.s. and thus Q_{n_0} -a.s., where $\widetilde{M} := M / \exp(\langle L \rangle^{\tau_{n_0}})$ with $\langle L \rangle^{\tau_{n_0}}(\cdot) := \langle L \rangle(\tau_{n_0} \wedge \cdot)$ is a Q_{n_0} -local martingale by Girsanov's theorem. Markov's inequality then implies that $Q_{n_0}(\tau_{n_0} \leq t) \leq \epsilon$ and an application of Corollary 1 concludes. \square

The next observation is useful when applying Theorem 1 in a continuous setup:

Lemma 1. *With the notation of Theorem 1, let $\{L_n\}_{n \in \mathbb{N}}$ denote a sequence of stochastic processes, with L_n a continuous Q_n -local martingales with quadratic variation $\langle L_n \rangle$ and assume that the sequence $\{\langle L_n \rangle(t)\}_{n \in \mathbb{N}}$ is tight along the sequence $\{Q_n\}_{n \in \mathbb{N}}$ of probability measures for some $t \in [0, \infty]$. Then also the sequence $\{L_n(t)\}_{n \in \mathbb{N}}$ is tight along $\{Q_n\}_{n \in \mathbb{N}}$.*

Proof. Fix $n \in \mathbb{N}$, let ρ_κ denote the first hitting time of κ by $\langle L_n \rangle$, and observe that

$$\begin{aligned} Q_n(L_n(t) > \kappa) &\leq Q_n(L_n(t \wedge \rho_\kappa) > \kappa) + Q_n(\rho_\kappa \leq t) \leq \frac{E_n[L_n^2(t \wedge \rho_\kappa)]}{\kappa^2} + Q_n(\rho_\kappa \leq t) \\ &\leq \frac{1}{\kappa} + Q_n(\langle L_n \rangle(t) > \kappa) \end{aligned}$$

for all $\kappa > 0$ by Chebyshev's inequality and the fact that $E_n[L_n^2(t \wedge \rho_\kappa)] \leq E_n[\langle L_n \rangle(t \wedge \rho_\kappa)] \leq \kappa$ due to the boundedness from below and local martingality of $L_n^2(\cdot \wedge \rho_\kappa) - \langle L_n \rangle(\cdot \wedge \rho_\kappa)$. This yields the statement. \square

3 Applications

Our goal here is to show that our approach could have advantages in terms of its relative simplicity.

Continuous processes

We begin by proving a well-known result by Beneš (1971) on the existence of weak solutions to a certain stochastic differential equation. We discuss it to illustrate how considerations of tightness as suggested here can often simplify the argument that a certain process is a martingale.

Proposition 2. *Fix $d \in \mathbb{N}$ and let $W = \{W(t)\}_{t \geq 0}$ be a d -dimensional Brownian motion, $W^* = \{W^*(t)\}_{t \geq 0}$ the running maximum of its vector norm; to wit, $W^*(t) := \max_{s \in [0, t]} \{\|W(s)\|_1\}$, and suppose that $\mu = \{\mu(t)\}_{t \geq 0}$ is a progressively measurable process satisfying*

$$\|\mu(t)\|_1 \leq c(1 + W^*(t)) \quad (2)$$

for some $c > 0$.

Then the local martingale $M = \{M(t)\}_{t \geq 0}$, defined as

$$M(t) := \exp \left(\int_0^t \mu(s) dW(s) - \frac{1}{2} \int_0^t \|\mu(s)\|_2^2 ds \right),$$

is a martingale.

Proof. Note that the sequence of simple progressively measurable processes $\mu_n = \{\mu_n(t)\}_{t \geq 0}$, defined by $\mu_n(\cdot) := (\mu(\cdot) \wedge n) \vee (-n)$, where the minimum and maximum are taken component by component, satisfies

$$\lim_{n \uparrow \infty} E \left[\int_0^t \|\mu_n(s)\|^2 ds \right] = E \left[\int_0^t \|\mu(s)\|^2 ds \right]$$

for all $t \geq 0$ by monotone convergence. By the definition of the stochastic integral, the sequence of local martingales $M_n = \{M_n(t)\}_{t \geq 0}$, defined as

$$M_n(t) := \exp \left(\int_0^t \mu_n(s) dW(s) - \frac{1}{2} \int_0^t \|\mu_n(s)\|_2^2 ds \right),$$

satisfies therefore $(P, M_n(t)) \xrightarrow{w} (P, M(t))$ for all $t \geq 0$. By Corollary 2, M_n is a true martingale since μ_n is bounded. Now, observe that

$$B_n(\cdot) := W(\cdot) + \int_0^\cdot \mu_n(s) ds$$

is Brownian motion under the probability measure Q_n , induced by $M_n(\cdot)$ via $dQ_n = M_n(t)dP$, and that

$$M_n(t) = \exp \left(\int_0^t \mu_n(s) dB_n(s) + \frac{1}{2} \int_0^t \|\mu_n(s)\|_2^2 ds \right). \quad (3)$$

We first note that

$$\|W(t)\|_1 \leq \|B_n(t)\|_1 + \int_0^t c(1 + W^*(s)) ds \leq B_n^*(t) + ct + c \int_0^t W^*(s) ds$$

for all $r \leq t$, where $B_n^* = \{B_n^*(t)\}_{t \geq 0}$ is defined similar to W^* . An application of Gronwall's inequality then yields that $W^*(t)$ is tight along $\{Q_n\}_{n \in \mathbb{N}}$. This guarantees that $\{\int_0^t \|\mu_n(s)\|_2^2 ds\}_{n \in \mathbb{N}}$ is tight as well. Lemma 1 then yields the tightness of $\{\int_0^t \mu_n(s) dB_n(s)\}_{n \in \mathbb{N}}$. Thus, $\{M_n(t)\}_{n \in \mathbb{N}}$ is tight along $\{Q_n\}_{n \in \mathbb{N}}$ and M is a true P -martingale by Theorem 1. \square

Now, suppose that $\tilde{\mu} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ is measurable and satisfies

$$\|\tilde{\mu}(t, y)\|_1 \leq c(1 + \|y\|_1)$$

for some $c > 0$. Then, with $\mu(t) = \tilde{\mu}(t, W(t))$ in the last proposition, the above computations show the weak existence of a solution to the stochastic differential equation

$$X(\cdot) = - \int_0^\cdot \tilde{\mu}(s, X(s)) ds + B(\cdot),$$

where $B = \{B(t)\}_{t \geq 0}$ denotes a Brownian motion.

Before proceeding we make the following simple observation: For a progressively measurable d -dimensional process $g = \{g(t)\}_{t \geq 0}$, define $\sigma(g) := \inf\{t \geq 0 : \int_0^t \|g(s)\|_2^2 ds = \infty\}$. Fix a d -dimensional Brownian motion $W = \{W(t)\}_{t \geq 0}$ and observe that the process $M = \{M(t)\}_{t \geq 0}$ with

$$M(t) := \exp\left(\int_0^t g(s) dW(s) - \frac{1}{2} \int_0^t \|g(s)\|_2^2 ds\right) := \exp\left(\int_0^t g(s) dW(s) - \frac{1}{2} \int_0^t \|g(s)\|_2^2 ds\right) \mathbf{1}_{\{\sigma(g) > t\}}$$

is well-defined, satisfies $M(t) = 0$ on $\{\sigma(g) \leq t\}$ and is a continuous local martingale; see also Lemma 1 in Ruf (2012).

We continue by formulating further necessary and sufficient conditions for a continuous local martingale to be a true martingale. Towards this end, we first introduce some notation. In the following, we fix an open set $E \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$ and denote by $C(\tilde{E})$ (a small extension of) the space of continuous paths $X : [0, \infty) \rightarrow \tilde{E}$ taking values in the one-point compactification \tilde{E} of E and getting absorbed in the cemetery state; see p. 40 in Pinsky (1995) for technical details. We denote the Borel-sigma algebra on $C(\tilde{E})$ by $\mathcal{F}^{\tilde{E}}$ and the filtration generated by the paths $X \in C(\tilde{E})$ by $\mathbb{F}^{\tilde{E}} := \{\mathcal{F}_t^{\tilde{E}}\}_{t \geq 0}$. For any $X \in C(\tilde{E})$, we write $\rho_E(X) := \inf\{t \geq 0 : X(t) \notin E\}$, which is a stopping time. Observe that $\mathcal{F}^{\tilde{E}} = \mathcal{F}_{\rho_E}^{\tilde{E}} = \bigvee_{j \in \mathbb{N}} \mathcal{F}_{\rho_{E_j}}^{\tilde{E}}$. We call a function g with domain $[0, \infty) \times C(\tilde{E})$ non-anticipating if for all $t \geq 0$ we have $g(t, X) = g(t, \tilde{X})$ for all X, \tilde{X} with $X|_{[0, t]} \equiv \tilde{X}|_{[0, t]}$.

The next definition is in the spirit of Section 1.13 in Pinsky (1995):

Definition 1 (Generalized local martingale problem). Given $d_1 \in \mathbb{N}$, an open set $E \subset \mathbb{R}^{d_1}$ of the form $E = \bigcup_{j \in \mathbb{N}} E_j$ for an increasing sequence $\{E_j\}_{j \in \mathbb{N}}$ of subsets of \mathbb{R}^{d_1} whose closure is contained in E , and $x_0 \in E$, let $\mu : [0, \infty) \times C(\tilde{E}) \rightarrow \mathbb{R}^{d_1}$ and $a : [0, \infty) \times C(\tilde{E}) \rightarrow \mathbb{R}^{d_1 \times d_1}$ be non-anticipating functions and $a(t, X)$ be symmetric and non-negative definite for all $(t, X) \in [0, \infty) \times C(\tilde{E})$. We call a probability measure P on $(C(\tilde{E}), \mathcal{F}^{\tilde{E}}, \mathbb{F}^{\tilde{E}})$ a solution to the corresponding generalized local martingale problem if $P(X(0) = x_0) = 1$ and

$$f(X(\cdot \wedge \rho_{E_j})) - \int_0^{\cdot \wedge \rho_{E_j}} \left(\sum_{i=1}^{d_1} \mu_i(s, X) f_{x_i}(X(s)) + \frac{1}{2} \sum_{i,j=1}^{d_1} a_{i,j}(s, X) f_{x_i, x_j}(X(s)) \right) ds \quad (4)$$

is a P -local martingale for all $j \in \mathbb{N}$ and $f \in C^2(E)$, the space of twice differentiable functions defined on E , with partial derivatives f_{x_i} and f_{x_i, x_j} . \square

In the setup of the last definition, fix $d_2 \in \mathbb{N}$ and consider a non-anticipative function $\sigma : [0, \infty) \times C(\tilde{E}) \rightarrow \mathbb{R}^{d_1 \times d_2}$ satisfying $a = \sigma \sigma^\top$. If the corresponding generalized local martingale problem has a solution P then, similarly as in Proposition 5.4.6 of Karatzas and Shreve (1991),

there exists a d_2 -dimensional Brownian motion $W^P = \{W^P(t)\}_{t \geq 0}$ on an extension of $(C(\tilde{E}), \mathcal{F}^{\tilde{E}}, P)$ such that

$$X(\cdot \wedge \rho_{E_j}) = x_0 + \int_0^{\cdot \wedge \rho_{E_j}} \mu(s, X) ds + \int_0^{\cdot \wedge \rho_{E_j}} \sigma(s, X) dW^P(s) \quad (5)$$

for all $j \in \mathbb{N}$.

Proposition 3. Fix $d_1, d_2 \in \mathbb{N}$, an open set $E \subset \mathbb{R}^{d_1}$ of the form $E = \cup_{j \in \mathbb{N}} E_j$ for an increasing sequence $\{E_j\}_{j \in \mathbb{N}}$ of subsets of \mathbb{R}^{d_1} whose closure is contained in E , and $x_0 \in E$. Let $\mu : [0, \infty) \times C(E) \rightarrow \mathbb{R}^{d_1}$, $b : [0, \infty) \times C(E) \rightarrow \mathbb{R}^{d_2}$ and $\sigma : [0, \infty) \times C(E) \rightarrow \mathbb{R}^{d_1 \times d_2}$ be non-anticipating such that

$$\int_0^t \|b(s, \omega)\|_2^2 \mathbf{1}_{\{\rho_{E_j}(\omega) > s\}} ds < \infty \quad (6)$$

for all $(t, j, \omega) \in (0, \infty) \times \mathbb{N} \times C(E)$ and set $a = \sigma \sigma^\top$.

Assume that the corresponding generalized martingale problem has a solution P and define the local martingale $M = \{M(t)\}_{t \geq 0}$ by

$$M(t) := \exp \left(\int_0^t b(s, X) dW^P(s) - \frac{1}{2} \int_0^t \|b(s, X)\|_2^2 ds \right), \quad (7)$$

where $W^P = \{W^P(t)\}_{t \geq 0}$ denotes the d_2 -dimensional Brownian motion of (5).

Then, $M^{\rho_{E_j}}$ is a uniformly integrable martingale for all $j \in \mathbb{N}$, and the corresponding generalized local martingale problem with $\mu(\cdot, \cdot)$ in (4) replaced by $\mu(\cdot, \cdot) + \sigma(\cdot, \cdot)b(\cdot, \cdot)$ has a solution Q such that $X(\cdot \wedge \rho_{E_j})$ has the same distribution under Q as under the measure Q_j defined by $dQ_j = M^{\rho_{E_j}}(\infty) dP$ for all $j \in \mathbb{N}$. Furthermore, Q is the unique solution such that $X(\cdot \wedge \rho_{E_j})$ has the same distribution under Q as under the measure Q_j for all $j \in \mathbb{N}$.

Proof. Fix $j \in \mathbb{N}$ and let $\{\tau_n\}_{n \in \mathbb{N}}$ denote the hitting times of $n \in \mathbb{N}$ by $M^{\rho_{E_j}}$. Define the sequence of uniformly integrable martingales $\{M_n\}_{n \in \mathbb{N}}$ by $M_n := M^{\rho_{E_j} \wedge \tau_n}$ and the probability measures \tilde{Q}_n by $d\tilde{Q}_n = M_n(\infty) dP$. By the second part of Theorem 1 it is sufficient to observe that $\{M_n(\infty)\}_{n \in \mathbb{N}}$ is tight along $\{\tilde{Q}_n\}_{n \in \mathbb{N}}$ due to (6) and Lemma 1 in order to obtain that $M^{\rho_{E_j}}$ is a uniformly integrable martingale.

Observe now that $Q_i(A) = Q_j(A)$ for all $A \in \mathcal{F}_{\rho_{E_i}}^{\tilde{E}}$ and $i, j \in \mathbb{N}$ with $i \leq j$. Thus, the set function Q , defined on $\cup_{j \in \mathbb{N}} \mathcal{F}_{\rho_{E_j}}^{\tilde{E}}$ by $Q(A) := Q_j(A)$ for $A \in \mathcal{F}_{\rho_{E_j}}^{\tilde{E}}$, is well-defined. By Lenglart's extension of Girsanov's theorem (see for example Theorem VIII.1.4 in Revuz and Yor, 1999) it remains to show that Q can be extended to a probability measure on $\mathcal{F}^{\tilde{E}} = \bigvee_{j \in \mathbb{N}} \mathcal{F}_{\rho_{E_j}}^{\tilde{E}}$. However, this follows from a standard extension theorem, such as Theorem V.4.1 in Parthasarathy (1967), whose conditions are easy to check; see Exercise 1.11 in Pinsky (1995) and Remark 6.1.1 in Föllmer (1972). The asserted uniqueness of Q follows from the fact that $\cup_{j \in \mathbb{N}} \mathcal{F}_{\rho_{E_j}}^{\tilde{E}}$ generates $\mathcal{F}^{\tilde{E}}$. \square

We now are ready to state a characterization, which generalizes the observations in Rydberg (1997). The following theorem shows that the existence and uniqueness of a weak solution to a certain stochastic differential equation yields the martingality of a related local martingale. We

remark that for the one-dimensional, time-homogeneous case, stronger results have been obtained; see Mijatović and Urusov (2012) and the references therein.¹

Theorem 2. *Assume the setup and notation of Proposition 3. We then have the following equivalences:*

1. $\int_0^t \|b(s, X)\|_2^2 ds < \infty$ Q -a.s. for all $t \geq 0$ if and only if M is a martingale;
2. $\int_0^\infty \|b(s, X)\|_2^2 ds < \infty$ Q -a.s. if and only if M is a uniformly integrable martingale.

Proof. For the first equivalence, start by assuming that M is a martingale. Then we need to show that $Q(A_n) = 0$ for the sequence of increasing events $\{A_n\}_{n \in \mathbb{N}}$ defined by

$$A_n := \left\{ \int_0^n \|b(s, X)\|_2^2 ds = \infty \right\}$$

for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and observe that the martingality of M yields a measure Q^M defined by $dQ^M = M(n)dP$. Since $\mathcal{F}_{\tau_n}^{\tilde{E}} = \bigvee_{j \in \mathbb{N}} \mathcal{F}_{\rho_{E_j} \wedge \tau_n}^{\tilde{E}}$, it is easy to see that $Q^M|_{\mathcal{F}_n^{\tilde{E}}} = Q|_{\mathcal{F}_n^{\tilde{E}}}$. Thus, $Q(A_n) = Q^M(A_n) = E^P[M(n)\mathbf{1}_{A_n}] = 0$ since $M(n) = 0$ P -a.s. on A_n .

For the reverse direction, let $\{\tau_n\}_{n \in \mathbb{N}}$ be a localization sequence of the local martingale M and define $M_n = M^{\tau_n}$. Fix $t > 0$. By Theorem 1 it is sufficient to show that $\{M_n(t)\}_{n \in \mathbb{N}}$ is tight along the sequence of probability measures $\{\tilde{Q}_n\}_{n \in \mathbb{N}}$, defined via $d\tilde{Q}_n = M_n(t)dP$. Then, the martingality of M follows directly. Lemma 1 yields that it is sufficient to show that $\{\int_0^{\tau_n \wedge t} \|b(s, X)\|_2^2 ds\}_{n \in \mathbb{N}}$ is tight. However, this follows from our assumption and the fact that again $\tilde{Q}_n|_{\mathcal{F}_{\tau_n}^{\tilde{E}}} = Q|_{\mathcal{F}_{\tau_n}^{\tilde{E}}}$. Alternatively, this direction could also have been proven by applying Lemma III.3.3 in Jacod and Shiryaev (2003).

The second equivalence follows in the same way. □

It is worth to ponder the implications of the previous result. It says that if one has two stochastic differential equations, each of which has a unique weak solution, which share the same functional form of the diffusion coefficient, and the solutions have the same support in path space, then one can always define the change of measure which is given by the corresponding putative exponential local martingale, which consequently is a true martingale. So, to show the martingality of the putative exponential local martingale, one can take advantage of the well developed theory for existence and uniqueness of stochastic differential equations.

Compound Poisson processes

We now continue with an application to compound Poisson processes. Towards this end, we denote the space of cadlag paths $\omega : [0, \infty) \rightarrow \mathbb{R}^d$ for some $d \in \mathbb{N}$ by D . Similar to above, we call a function g with domain $[0, \infty) \times D$ non-anticipating if for all $t \geq 0$ we have $g(t, \omega) = g(t, \tilde{\omega})$ for all $\omega, \tilde{\omega}$ with $\omega|_{[0, t]} \equiv \tilde{\omega}|_{[0, t]}$.

Now, fix $x_0 \in \mathbb{R}^d$ and a non-anticipating function $r : [0, \infty) \times D \rightarrow \mathbb{R}$. Let $N = \{N(t)\}_{t \geq 0}$ denote a Poisson process with unit rate and $Z = \{Z_j\}_{j \geq 1}$ a sequence of d -dimensional i.i.d. random

¹Mijatović and Urusov (2012) provide conditions in terms of the behavior of X under P and Q at the boundary points of the one-dimensional interval E . Using Feller's test of explosions they formulate these conditions using the scale functions of the corresponding diffusions. Furthermore, they do not assume the equivalent of (6). However, observe that often this assumption can be easily satisfied by modifying E . Moreover, we used the assumption only in the proof of Proposition 3 but not in the one of Theorem 2.

variables with $Z_1 \neq 0$. We say that a filtered probability space along with an adapted process X with cadlag paths is a weak solution to the stochastic differential equation

$$X(t) = x_0 + \sum_{j=1}^{L_X(t)} Z_j, \quad L_X(t) := N(\Lambda_X(t)), \quad \Lambda_X(t) := \int_0^t \exp(r(s, X)) ds \quad (8)$$

if $X(0) = x_0$,

$$\Lambda_X(t) = \int_0^t \exp(r(s, X)) ds < \infty$$

for all $t \geq 0$, $\sum_{s \leq \cdot} \mathbf{1}_{\{\Delta X(s) \neq 0\}}$ has the same distribution as $N(\Lambda_X(\cdot))$ and the jumps $\{\Delta X | \Delta X \neq 0\}$ of X are independent and identically distributed as Z_1 , where $\Delta X(t) := X(t) - X(t-)$. Thus, X corresponds to a compound Poisson process with jump sizes $\{Z_j\}_{j \in \mathbb{N}}$ such that its instantaneous intensity to jump at time t equals $\exp(r(t-, X))$.

It is clear that such a solution exists, for example, if $r(t, X) = \mathfrak{r}(t)$ only depends on time and $\int_0^t \exp(\mathfrak{r}(s)) ds < \infty$ for all $t \geq 0$. The following lemma gives another existence result:

Lemma 2. *Fix $d \in \mathbb{N}$ and $x_0 \in \mathbb{R}^d$. Let $\mathfrak{r} : \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable, such that there exists $c > 0$ with $\exp(\mathfrak{r}(y)) \leq c(1 + y)$ for all $y \in \mathbb{R}^d$, and let $N = \{N(t)\}_{t \geq 0}$ denote a Poisson process with unit rate and $Z = \{Z_j\}_{j \geq 1}$ a sequence of d -dimensional i.i.d. random variables, independent of N , with $E[\|Z_1\|_1] < \infty$ and $Z_1 \neq 0$. Then, there exists a weak solution to (8) with $r(t, \omega) = \mathfrak{r}(\omega(t))$ for all $(t, \omega) \in [0, \infty) \times D$.*

Proof. First observe that

$$J(t) := x_0 + \sum_{j=1}^{N(t)} Z_j$$

always exists and that

$$\Gamma(t) := \int_0^t \exp(-\mathfrak{r}(J(s))) ds \geq \frac{1}{c} \int_0^t \frac{1}{1 + \|J(s)\|_1} ds$$

is strictly increasing and tends to infinity since there exists $K(\omega) \in (0, \infty)$ such that $\|J(t)\|_1 \leq K(\omega)(1 + t)$ by the law of large numbers. Thus, Γ yields a valid time change. Now, consider $X(t) = J(\Gamma^{-1}(t))$ and observe that

$$\Gamma^{-1}(t) = \int_0^t \exp(\mathfrak{r}(J(\Gamma^{-1}(s)))) ds = \int_0^t \exp(\mathfrak{r}(X(s))) ds,$$

which yields that X is a solution to (8). \square

The next theorem gives a sufficient condition that guarantees that the intensity in Poisson processes can be changed without changing the nullsets of the underlying probability measure. For example, any compound Poisson process with a non-explosive and strictly positive intensity can be changed, via an equivalent change of measure, to a compound Poisson process with unit intensity (set $r_2 \equiv 1$ below).

Theorem 3. *Fix $d \in \mathbb{N}$ and $x_0 \in \mathbb{R}^d$. Let $r_1, r_2 : [0, \infty) \times D \rightarrow \mathbb{R}$ be non-anticipating and $N = \{N(t)\}_{t \geq 0}$ denote a Poisson process with unit rate and $Z = \{Z_j\}_{j \geq 1}$ a sequence of d -dimensional i.i.d. random variables independent of N .*

Consider the two stochastic differential equations

$$X^{(i)}(t) = x_0 + \sum_{j=1}^{L_{X^{(i)}}(t)} Z_j, \quad L_{X^{(i)}}(t) := N(\Lambda_{X^{(i)}}(t)), \quad \Lambda_{X^{(i)}}(t) := \int_0^t \exp(r_i(s, X^{(i)})) ds$$

for $i = 1, 2$ and assume that both equations have a weak solution, uniquely determined in distribution.

Then, the process $M = \{M(t)\}_{t \geq 0}$, defined by

$$M(t) := \exp \left(\int_0^t (r_2(s-, X^{(1)}) - r_1(s-, X^{(1)})) dL_{X^{(1)}}(s) - \int_0^t (\exp(r_2(s, X^{(1)})) - \exp(r_1(s, X^{(1)}))) ds \right),$$

is a true martingale; furthermore, under Q , defined on $\mathcal{F}(t)$ by $dQ|_{\mathcal{F}(t)} = M(t)dP|_{\mathcal{F}(t)}$, the distribution of $X^{(1)}$ equals the distribution of $X^{(2)}$ under P .

Proof. Theorem VI.2 in Brémaud (1981) yields that M is a local martingale. If M is a true martingale, then Theorem VI.3 in Brémaud (1981) yields the assertion on the distribution of $X^{(1)}$ under its new generated measure. As in the proof of Theorem 2, let $\{\tau_n\}_{n \in \mathbb{N}}$ be a localization sequence of M and define $M_n = M^{\tau_n}$. Fix $t > 0$. By Theorem 1 it is now sufficient to show that $\{M_n(t)\}_{n \in \mathbb{N}}$ is tight along the sequence of probability measures $\{Q_n\}_{n \in \mathbb{N}}$, defined via $dQ_n = M_n(t)dP$ to obtain the martingality of M . For $i = 1, 2$, we shall see that

$$\left\{ \int_0^{\tau_n \wedge t} |r_i(s-, X^{(1)})| dL_{X^{(1)}}(s) \right\}_{n \in \mathbb{N}} \quad \text{and} \quad \left\{ \int_0^{\tau_n \wedge t} \exp(r_i(s, X^{(1)})) ds \right\}_{n \in \mathbb{N}}$$

are tight along $\{Q_n\}_{n \in \mathbb{N}}$, which then proves the statement. Towards this end, Theorem VI.3 in Brémaud (1981) again yields that, up to the stopping time τ_n , the Q_n -dynamics of $X^{(1)}$ are exactly the P -dynamics of $X^{(2)}$. Thus, it is sufficient to observe that

$$\begin{aligned} Q_n \left(\int_0^{\tau_n \wedge t} |r_i(s-, X^{(1)})| dL_{X^{(1)}}(s) > \kappa \right) &= P \left(\int_0^{\tau_n \wedge t} |r_i(s-, X^{(2)})| dL_{X^{(2)}}(s) > \kappa \right) \\ &\leq P \left(\int_0^t |r_i(s-, X^{(2)})| dL_{X^{(2)}}(s) > \kappa \right) \end{aligned}$$

for all $\kappa > 0$, where the right-hand side does not depend on n and tends to zero as κ increases. The same observations hold for the other terms of M . \square

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